Entropy and mutual information in models of deep neural networks

NeurIPS 2018 - Thursday Dec 06th - Spotlight

Poster @ Room 210 & 230 AB #110

Marylou Gabrié (LPS ENS), Andre Manoel (INRIA Saclay, Owkin),
Clément Luneau, Jean Barbier, Nicolas Macris (EPFL),
Lenka Zdeborová (CEA Saclay), Florent Krzakala (LPS ENS)

Motivations

Information theoretic arguments to deep learning

- Theory of generalization: Shwartz-Ziv et al. 2017, Saxe et al. 2018, Goldfeld et al. 2018 etc.
- New regularizer: Chalk et al. 2016, Alemi et al. 2017, Kolchinsky et al. 2017, Belghazi et al. 2017, Zhao et al. 2018, Achille et al. 2018, Hjelm et al. 2018, etc.

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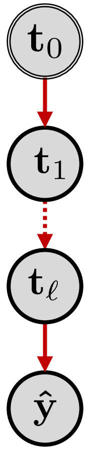
Computing mutual informations in neural networks …



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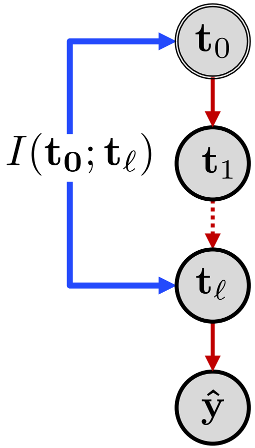

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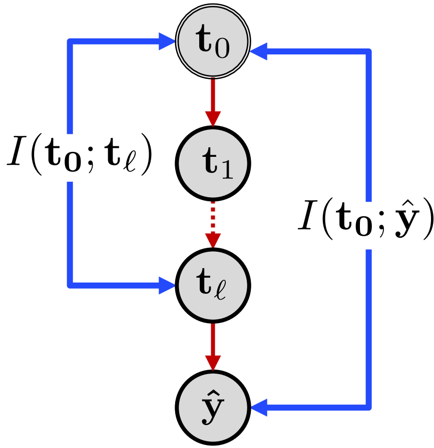

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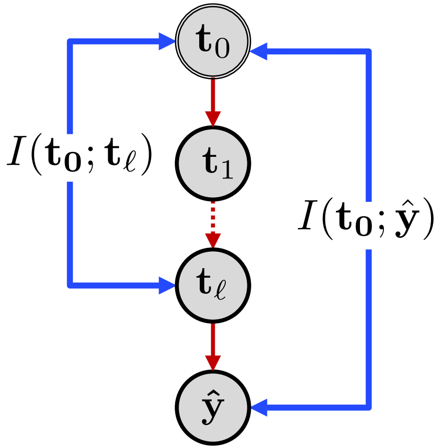
- Intractable $p(\mathbf{t}_\ell) = \int \mathrm{d}\mathbf{t}_0 \mathrm{d}\mathbf{t}_1 \cdots \mathrm{d}\mathbf{t}_{\ell-1}\, p(\mathbf{t}_\ell|\mathbf{t}_{\ell-1}) \cdots p(\mathbf{t}_1|\mathbf{t}_0) p(\mathbf{t}_0)$



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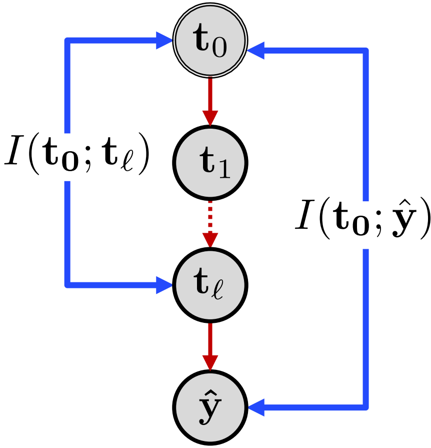

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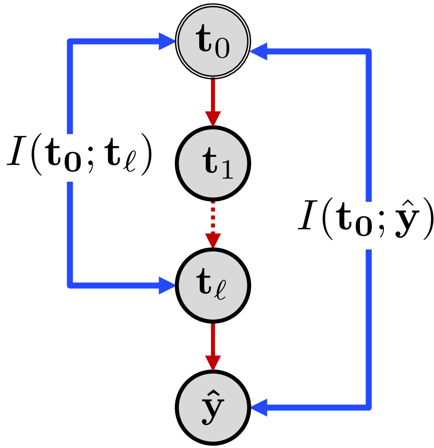

Sampled based estimations (non-parametric, variational methods)

Kraskov et al. 2014, Kolchinsky et al. 2017, Belghazi et al. 2017, Goldfeld et al. 2018 etc.

➔ Less and less reliable as networks get large

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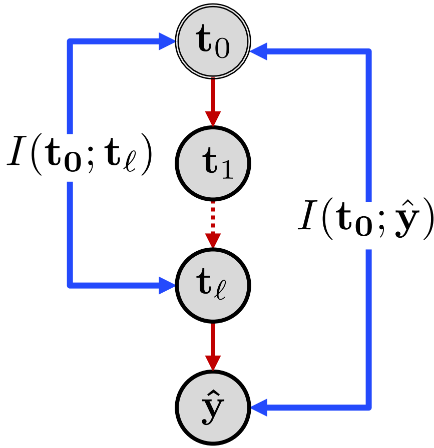

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Here 1) Restrict to **models** of DNNS, 2) Leverage statistical physics **replica method**

Model: fully connected feed forward neural network with stochastic activations

$$\overset{\mathbb{R}^{N_\ell}}{\mathbf{t}_\ell} = f_\epsilon(\mathbf{W}^{(\ell)} f_\epsilon(\mathbf{W}^{(\ell-1)} \cdots f_\epsilon(\mathbf{W}^{(1)} \overset{\mathbb{R}^{N_0}}{\mathbf{t}_0}) \cdots))$$

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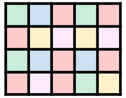

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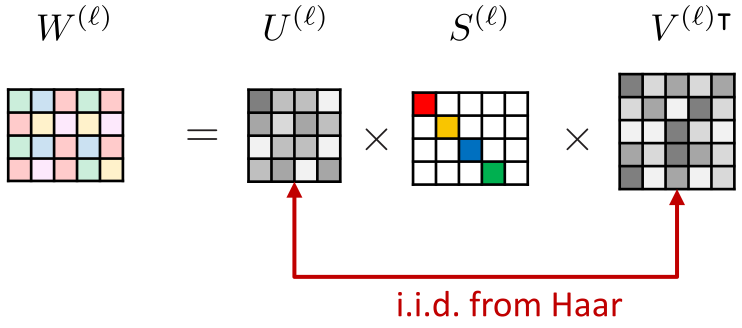

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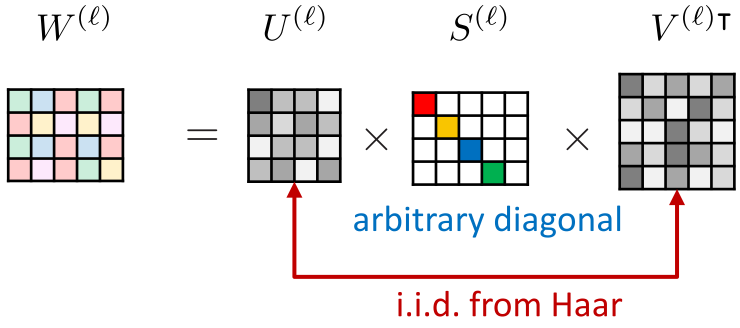

Main results

1) Formula: $\lim_{N_0\to\infty} \frac{1}{N_0} H(\mathbf{t}_\ell) = \min_{\mathbf{A},\mathbf{V},\tilde{\mathbf{A}},\tilde{\mathbf{V}}} \operatorname{extr} \phi_\ell(\mathbf{A},\mathbf{V},\tilde{\mathbf{A}},\tilde{\mathbf{V}})$

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2) Theorem:

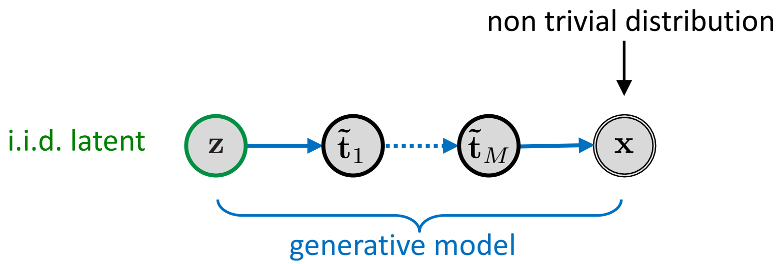
2-layer $\mathbf{t}_2 = f_\epsilon(\mathbf{W}^{(2)} f_\epsilon(\mathbf{W}^{(1)} \mathbf{t}_0))$
Gaussian random weight matrices

$\Rightarrow$ replica prediction exact in limit of large networks ✓

APPLICATION:
Follow mutual information during SGD learning

Applying the replica formula to learning experiments

1) Synthetic data sets for student



Keras implementation

marylou-gabrie / **learning-synthetic-data**

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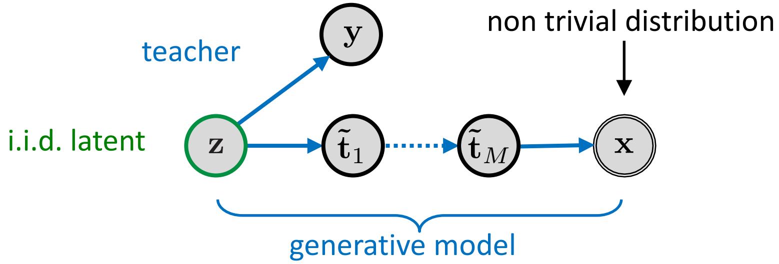

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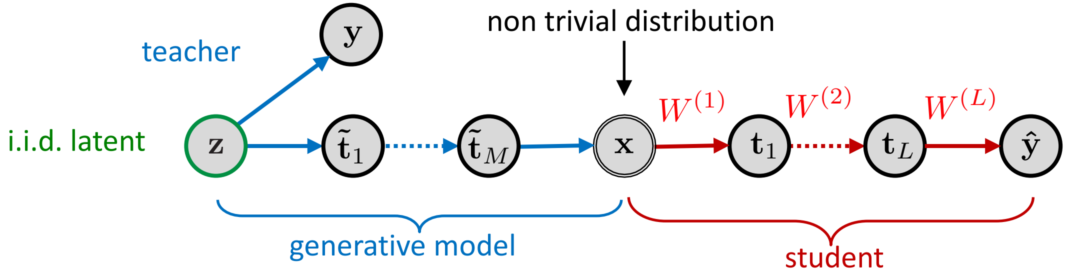

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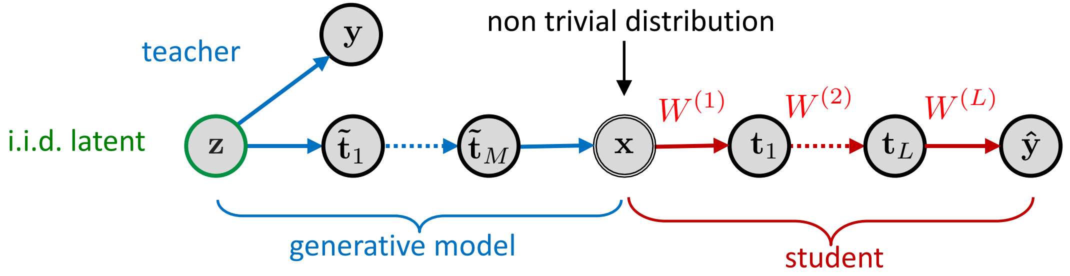
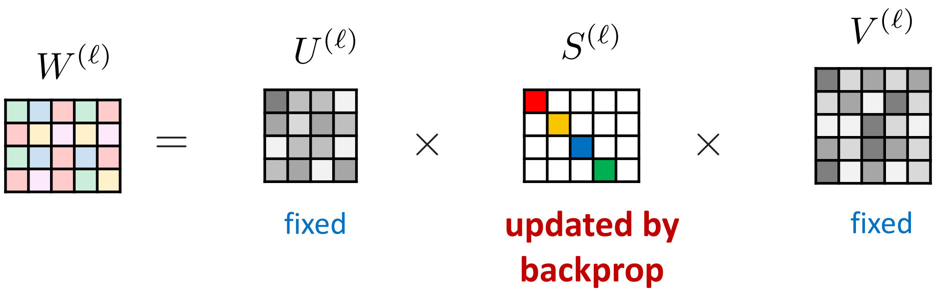


2) Orthogonally invariant weight matrices during learning for the student



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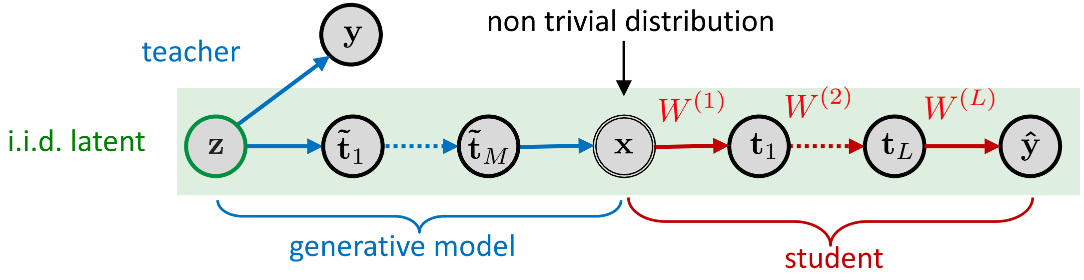

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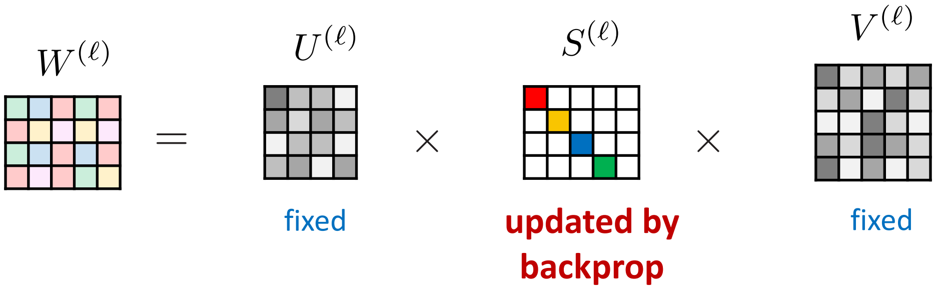

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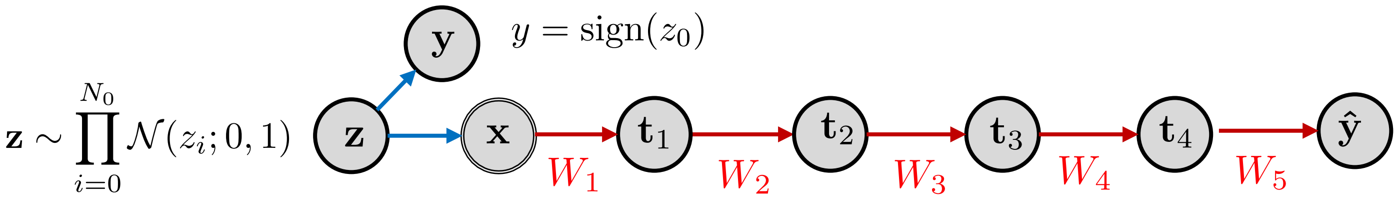
Exploratory experiment: A binary classification example



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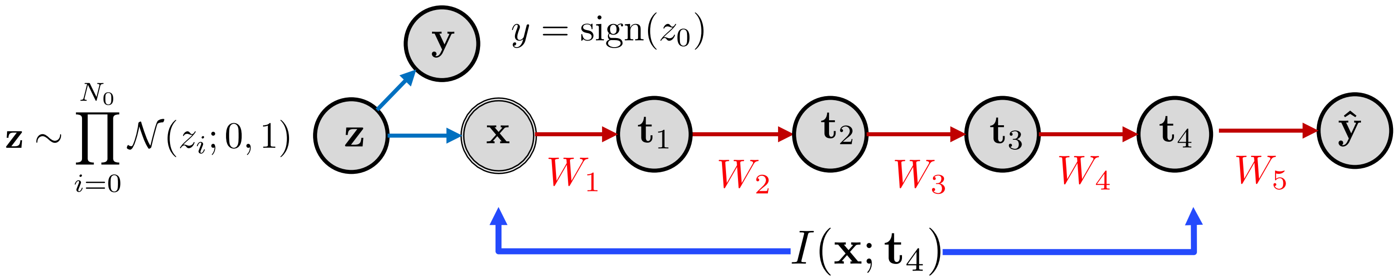

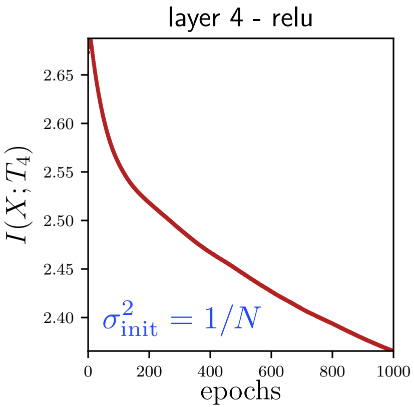

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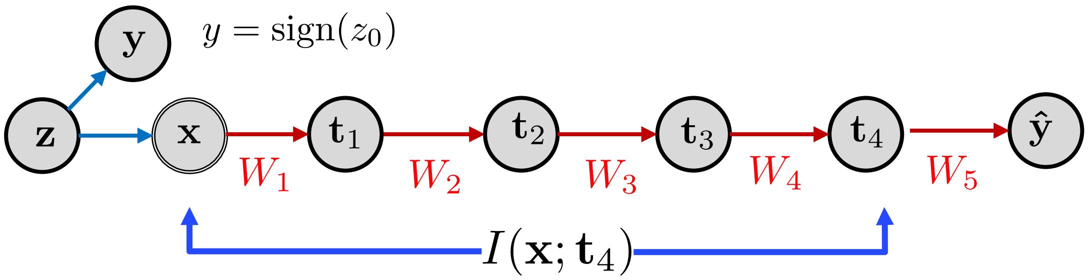
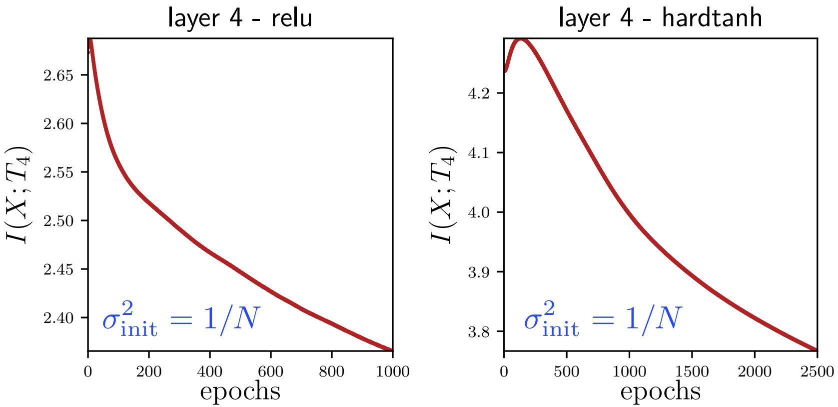
$$\mathbf{z} \sim \prod_{i=0}^{N_0} \mathcal{N}(z_i; 0, 1)$$



Different non-linearities



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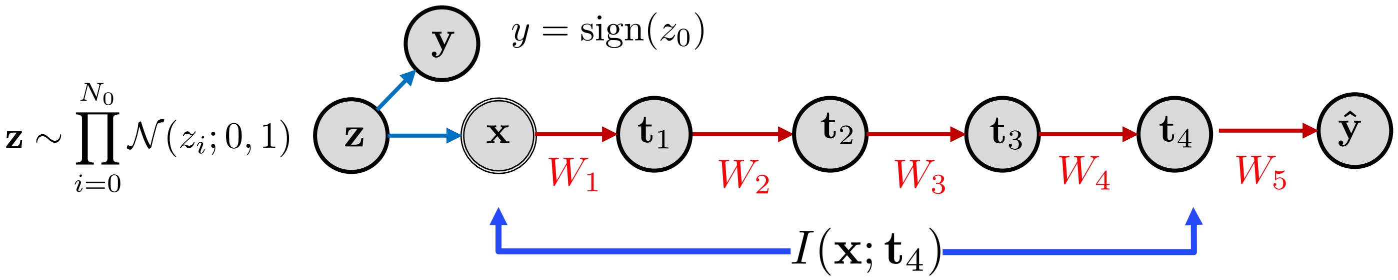

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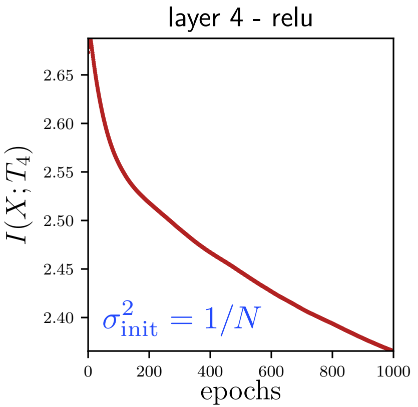
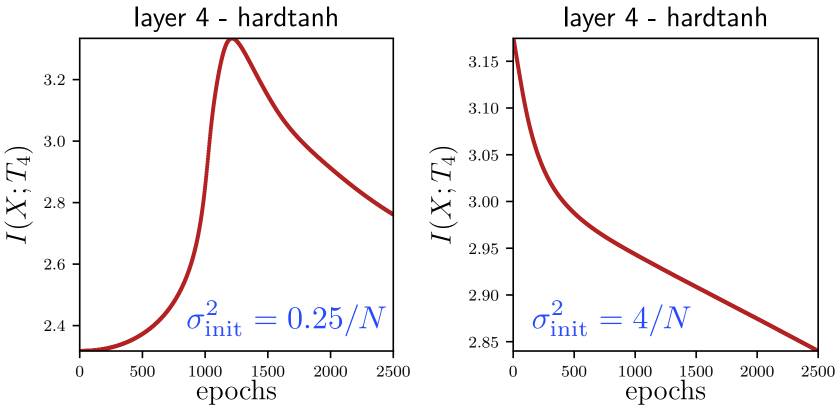


Different initializations



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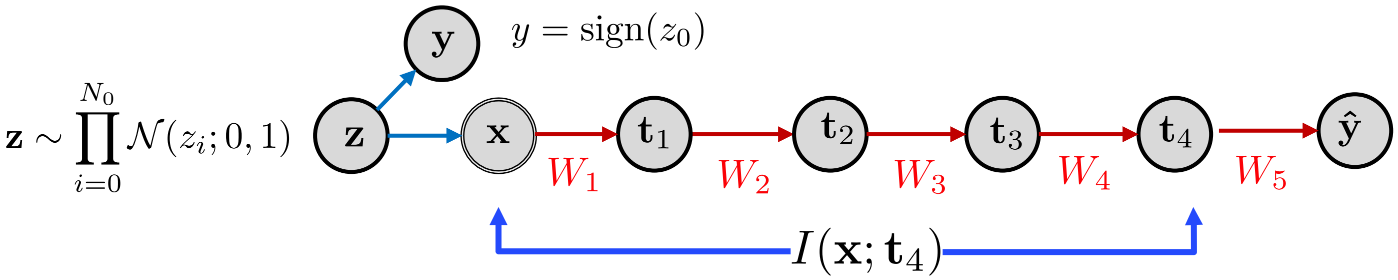

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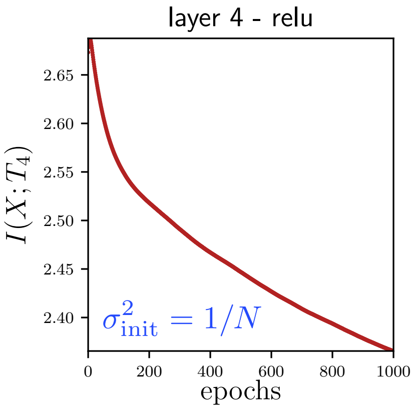

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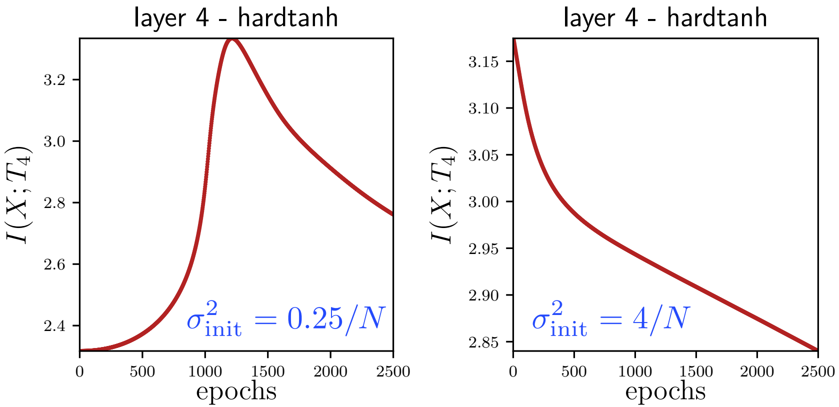

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