Structural Analysis of Branch-and-Cut and the Learnability of Gomory Mixed Integer Cuts

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Branch-and-bound

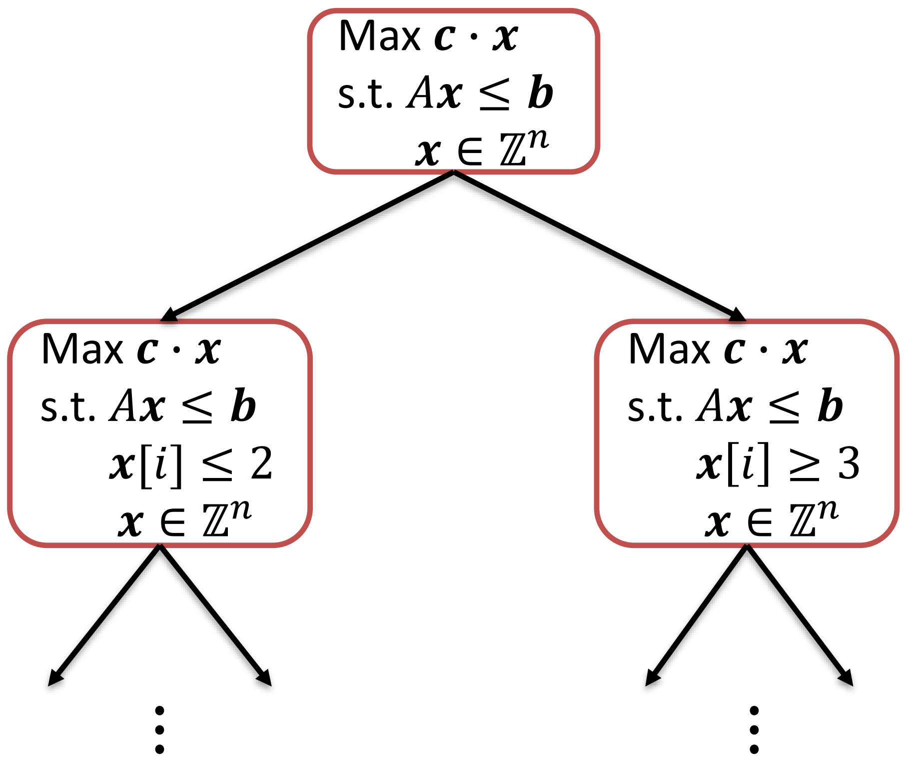
- Powerful tree-search algorithm used to solve IPs in practice

- Uses the linear programming (LP) relaxation to do an informed search through the set of feasible integer solutions

IP		LP relaxation
Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \in \mathbb{Z}^n$	→	Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \in \mathbb{R}^n$

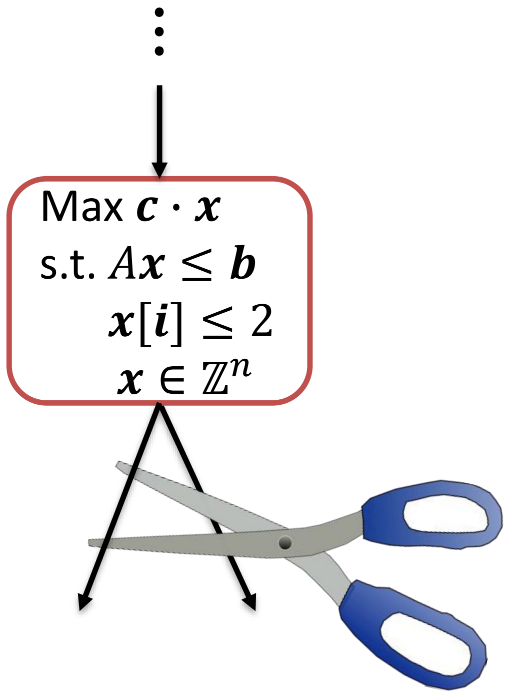
Branch-and-bound: branching

- Choose variable *i* to branch on.
- Generate one subproblem with $\boldsymbol{x}[i] \leq \lfloor \boldsymbol{x}_{\mathrm{LP}}^*[i] \rfloor$ another with $\boldsymbol{x}[i] \geq \lceil \boldsymbol{x}_{\mathrm{LP}}^*[i] \rceil$



Branch-and-bound: pruning

- Prune subtrees if
  - LP relaxation at a node is integral, infeasible, or
  - (Bounding) LP optimal *worse* than best feasible integer solution found so far

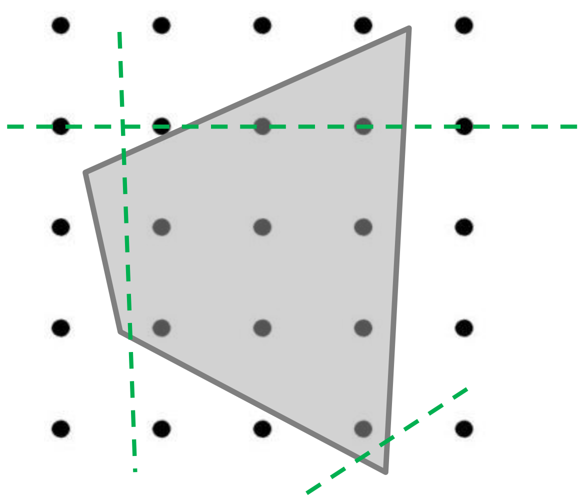


Branch-and-cut

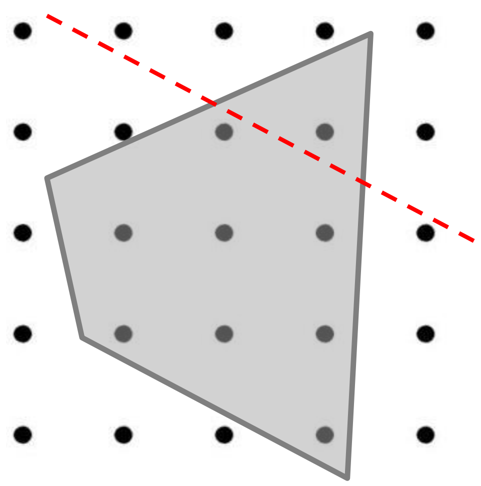
- Branch-and-bound, but at each node may add *cutting planes*
- Method of getting tighter LP relaxation bounds, and thus pruning subtrees sooner

Cutting planes

- Constraint $\boldsymbol{\alpha x} \leq \beta$ is a *valid cutting plane* if it does not cut off any integer feasible points



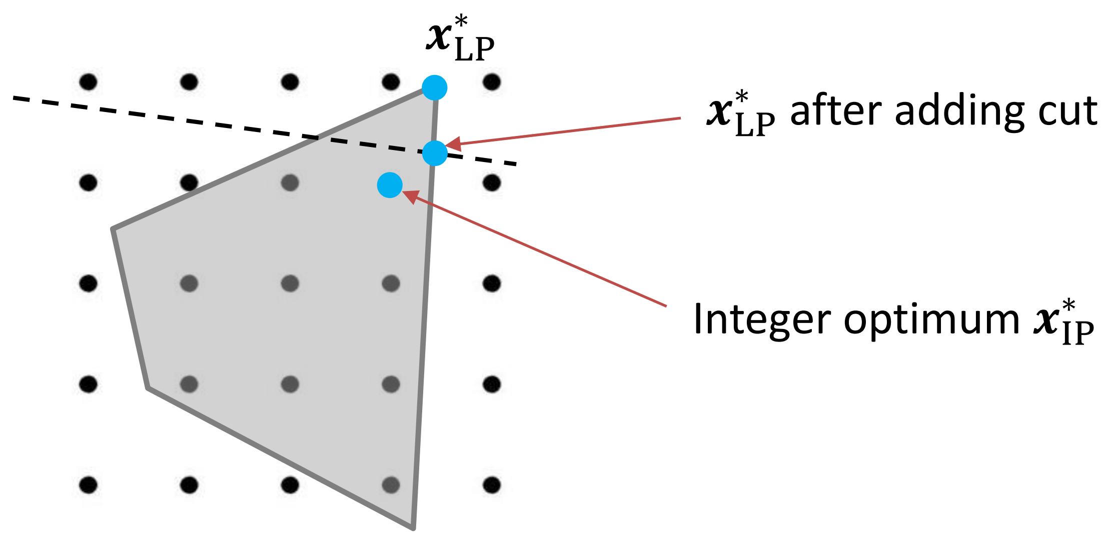
Valid cutting planes



An invalid cutting plane

Cutting planes

- If $\boldsymbol{\alpha x} \leq \beta$ is valid and separates the LP optimum, can speed up B&C by pruning nodes sooner



Gomory Mixed Integer Cuts

$$\sum_{i:f_i\le f_0} f_i x_i + \frac{f_0}{1-f_0}\sum_{i:f_i>f_0}(1-f_i)x_i \ge f_0$$

- *Gomory Cuts Revisited*, 1996. Balas, Ceria, Cornuéjols, Natraj
  - achieved remarkable speedups by effectively integrating these cuts into branch-and-bound (believed to be practically useless before)

- Today: Gomory mixed integer cuts a crucial component of commercial solvers like CPLEX and Gurobi

G. Cornuéjols. Revival of the Gomory cuts in the 1990’s. *Annals of OR*. 2007.

Main contributions

- The first generalization guarantees for using machine learning to add Gomory mixed integer cuts
- A novel structural analysis of the branch-and-cut algorithm that pins down its possible behaviors

Generalization guarantees for cutting planes

Distribution-dependent cut selection

Learning to cut

If a cut yields small average branch-and-cut tree size over IP samples…

$$\begin{array}{c}\text{Max } \boldsymbol{c}_{\mathbf{1}} \cdot \boldsymbol{x} \\ \text{s.t. } A_1 \boldsymbol{x} \leq \boldsymbol{b}_{\mathbf{1}} \\ \boldsymbol{x} \in \mathbb{Z}^n\end{array} \quad \bullet\bullet\bullet \quad \begin{array}{c}\text{Max } \boldsymbol{c}_N \cdot \boldsymbol{x} \\ \text{s.t. } A_N \boldsymbol{x} \leq \boldsymbol{b}_N \\ \boldsymbol{x} \in \mathbb{Z}^n\end{array} \quad \sim D$$

IP 1 IP N

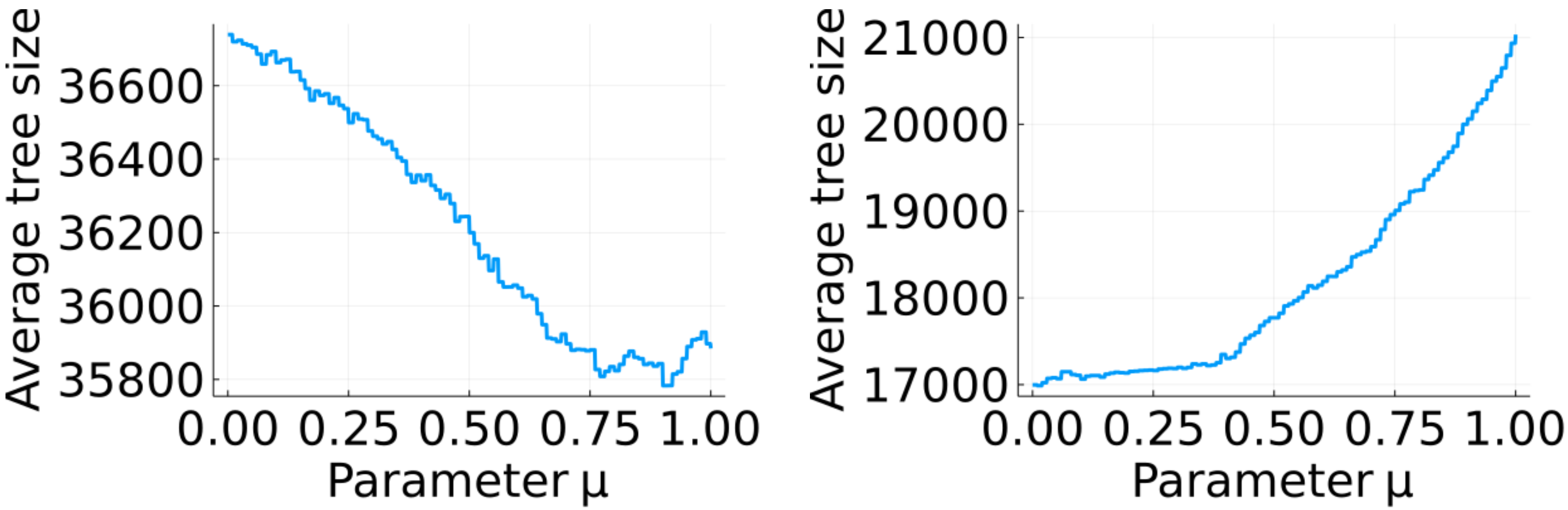
…is it likely to yield a small branch-and-cut tree on a fresh IP?

$$\begin{array}{c}\text{Max } \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t. } A \boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{x} \in \mathbb{Z}^n\end{array} \quad \sim D$$

Tuning a GMI cut selection parameter

- E.g. mixture of $d = 2$ scores

$$\mu \cdot \text{parallelism} + (1 - \mu) \cdot \text{efficacy}$$



Two different distributions over facility location IPs.

Generalization for Gomory mixed integer cuts

Theorem [Balcan, Prasad, Sandholm, Vitercik NeurIPS'22]: for all GMI cuts $\boldsymbol{u} \in [-U, U]^m$, difference between average training performance over N samples and expected performance is (whp)

$$\tilde{O}\left(H\sqrt{\frac{mn^3 \log\left(mn\tau U \|A\|_{1,1} \|\boldsymbol{b}\|_1\right)}{N}}\right)$$

Proof uses our structural analysis of branch-and-cut

A structural analysis of branch-and-cut

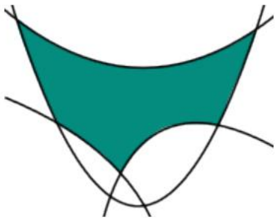
- Given two valid cutting planes

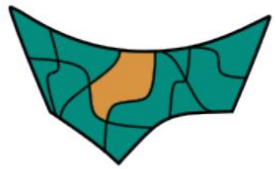
$$\boldsymbol{\alpha_1 x} \leq \beta_1 \text{ and } \boldsymbol{\alpha_2 x} \leq \beta_2$$

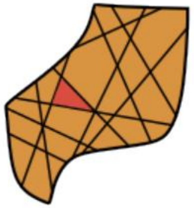
- When does B&C behave identically on the following IPs?

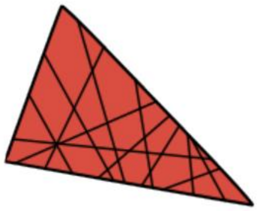
Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{\alpha_1 x} \leq \beta_1$ $\boldsymbol{x} \in \mathbb{Z}^n$	Max $\boldsymbol{c} \cdot \boldsymbol{x}$ s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{\alpha_2 x} \leq \beta_2$ $\boldsymbol{x} \in \mathbb{Z}^n$

Branch-and-cut piecewise invariance


(a) LP optimum closed form


(b) Invariant branching


(c) Invariant LP integrality


(d) Invariant B&C execution

Theorem [BPSV NeurIPS'22] : Given IP $(\boldsymbol{c}, A, \boldsymbol{b})$, there are $O\left(14^n(m+2n)^{3n^2}\tau^{5n^2}\right)$ polynomial hypersurfaces of degree ≤ 5 that partition $\mathbb{R}^{n+1}$ into connected components such that the branch-and-cut tree built after adding the cut $\boldsymbol{\alpha x} \leq \beta$ is invariant over all $(\boldsymbol{\alpha}, \beta)$ within a given component.

$$\tau := \left\lceil \max_{\boldsymbol{x} \in P} \|\boldsymbol{x}\|_\infty \right\rceil \leq \|A\|_{\infty,\infty}^n n^{n/2}$$

If A has a row with all positive entries, then $\tau \leq \|\boldsymbol{b}\|_\infty$