On the Ability of Graph Neural Networks to Model Interactions Between Vertices

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Joint work with Tom Verbin & Nadav Cohen

Tel Aviv University

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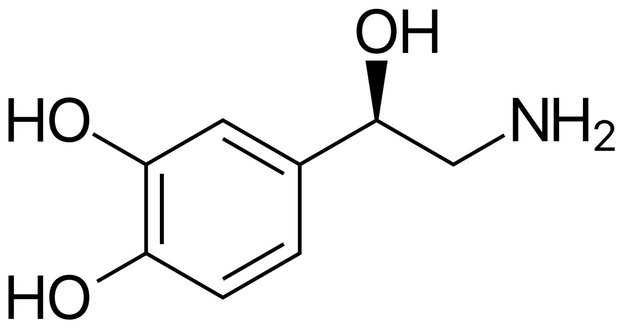
Graph Neural Networks (GNNs)

Neural networks purposed for **modeling interactions over graph data**



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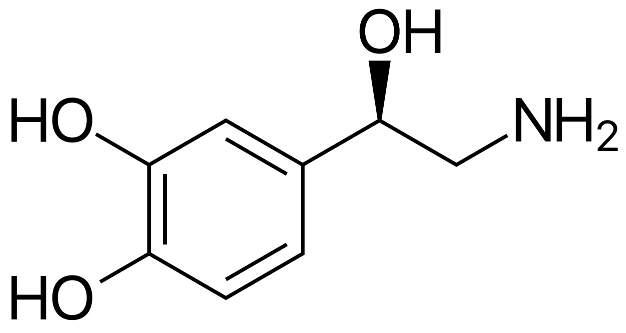
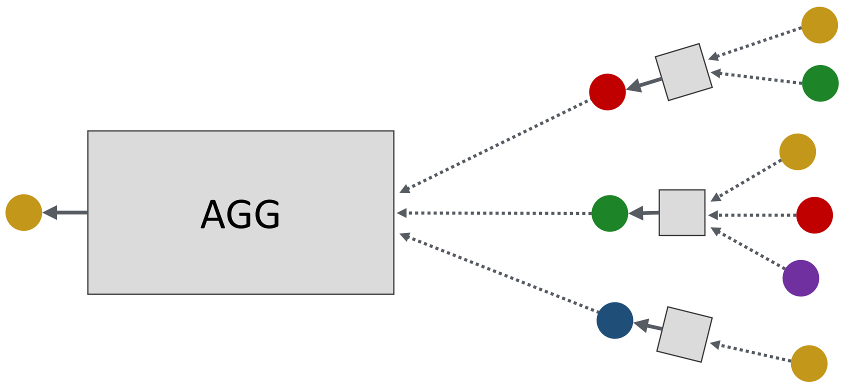


Vast majority of GNNs follow the **message-passing paradigm**



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Q: How do graph structure and GNN architecture affect modeled interactions?

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Widely used measure for the **interaction modeled across a partition of input variables**

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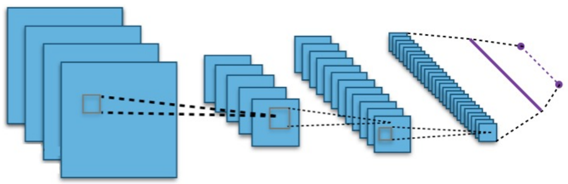
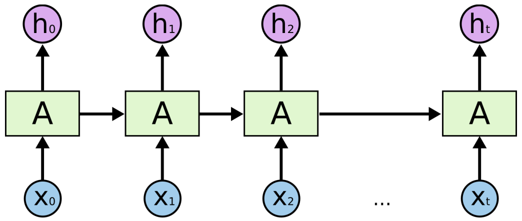
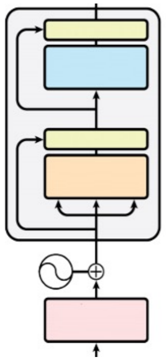
- Analyses of convolutional, recurrent, and self-attention NNs

  (e.g. Cohen & Shashua 2017, Levine et al. 2018;2020, **R** et al. 2022)







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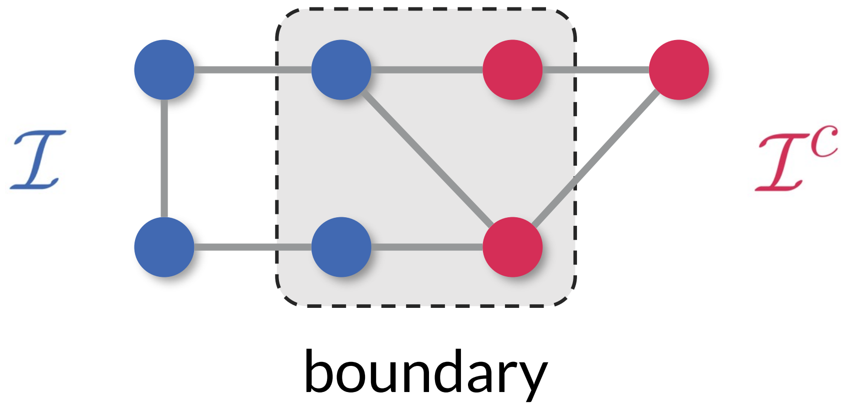

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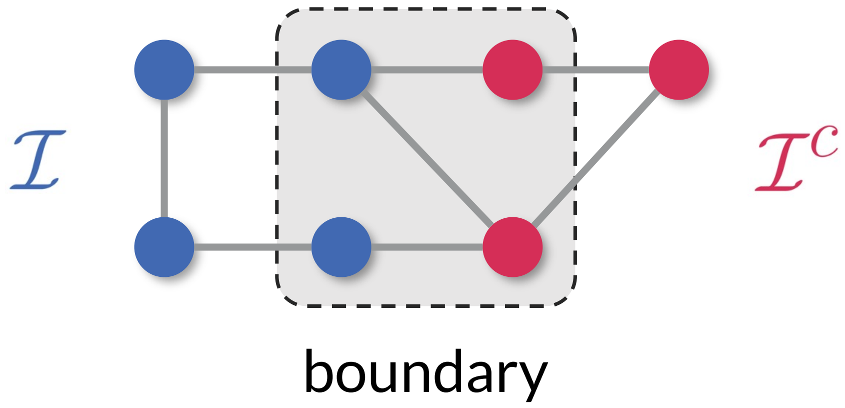

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$\mathrm{WI}(\mathcal{I}) :=$ # length $L-1$ walks from boundary

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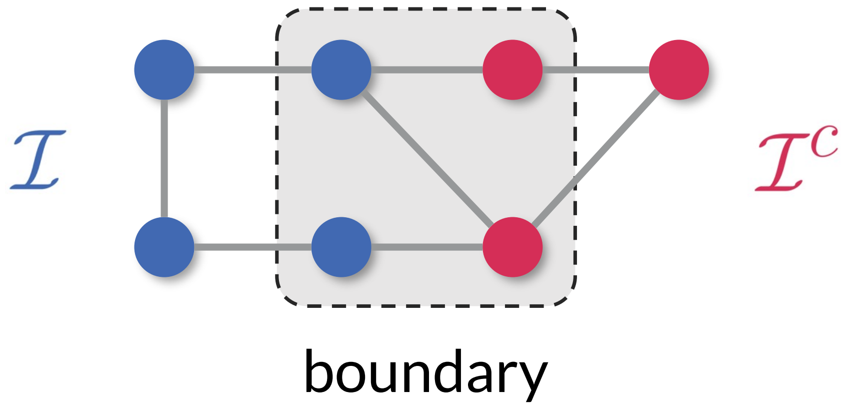

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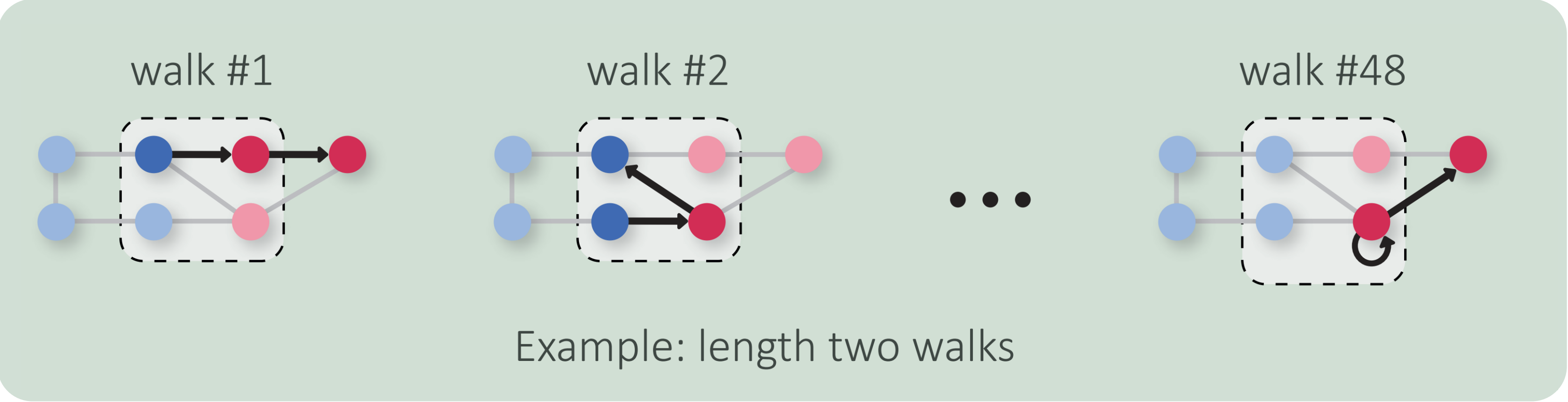

Main Result: Strength of Interaction ∝ Walk Index

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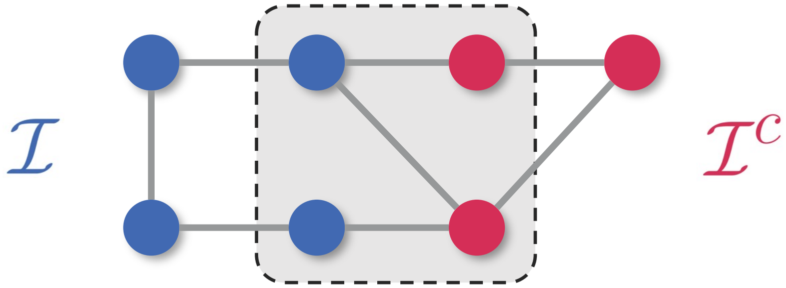

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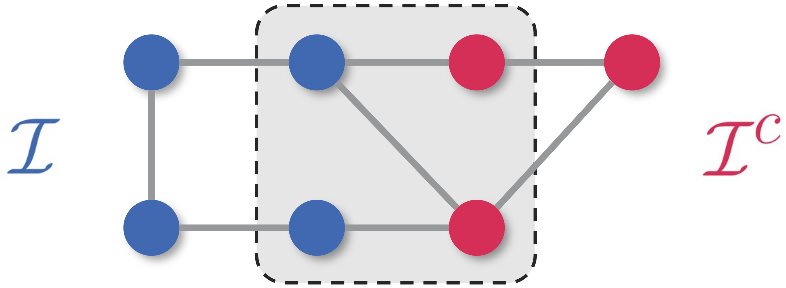

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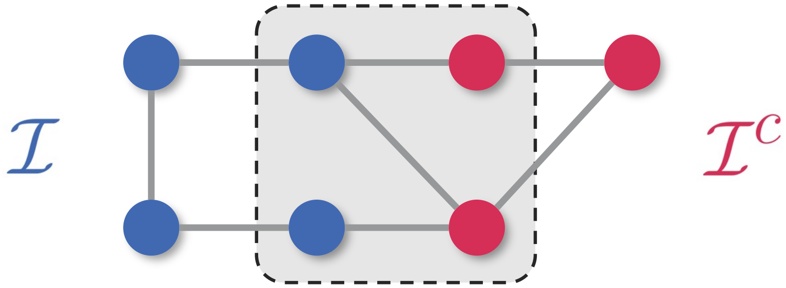

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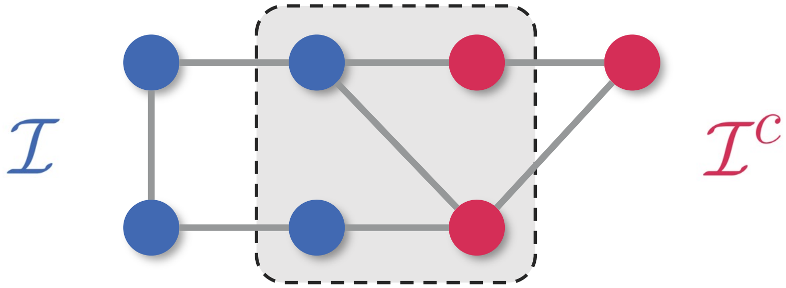

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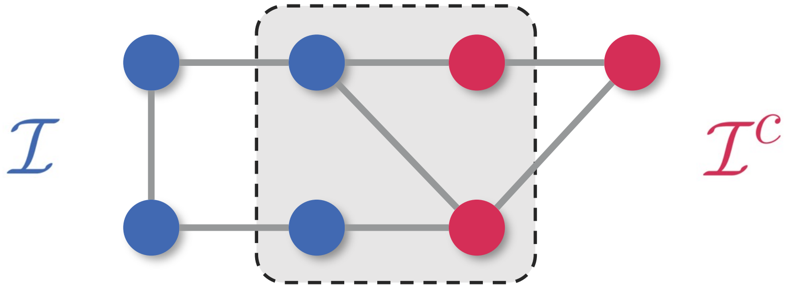

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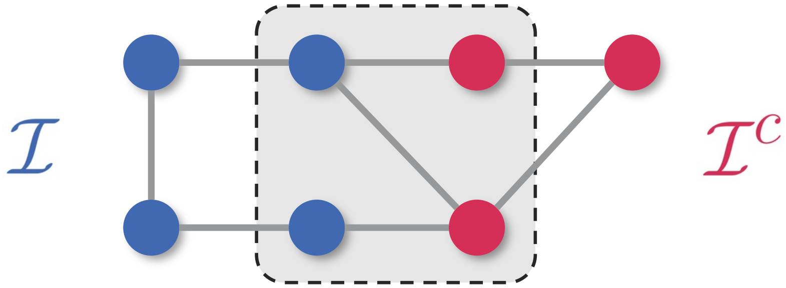

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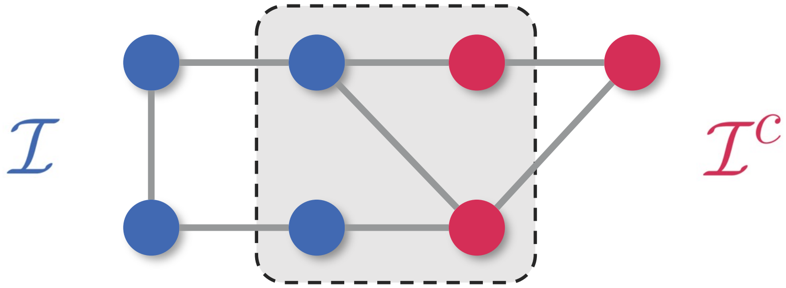

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Experiment: Implications of theory apply to GNNs with ReLU non-linearity (GCN, GAT, GIN)

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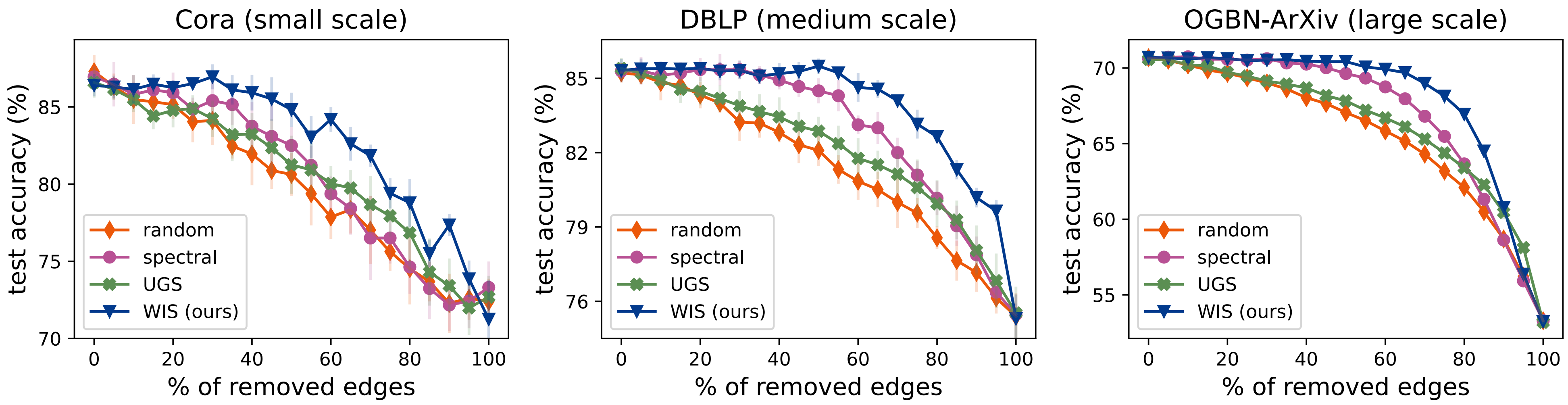

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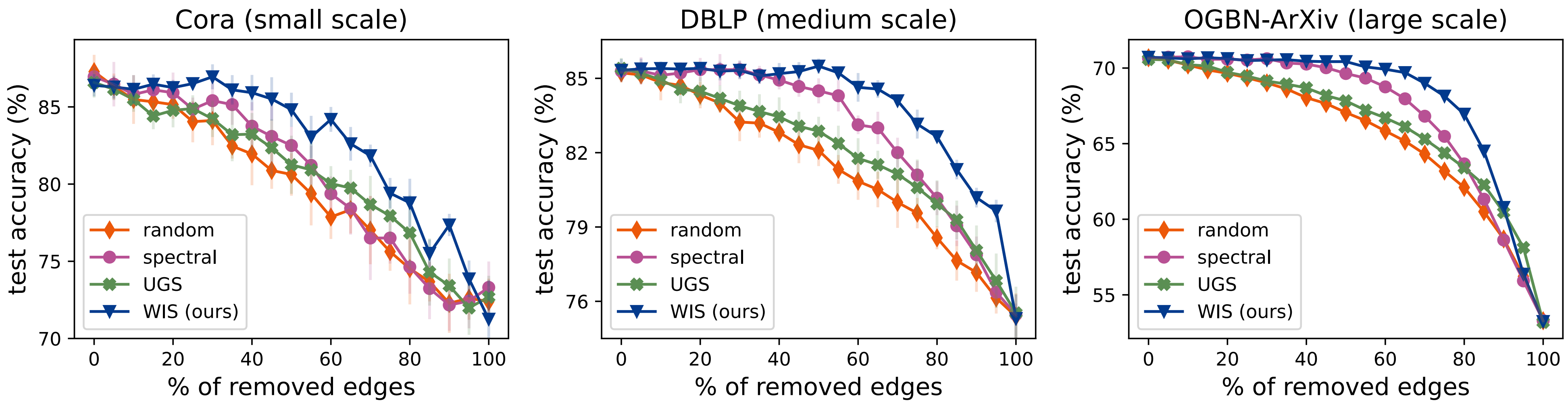

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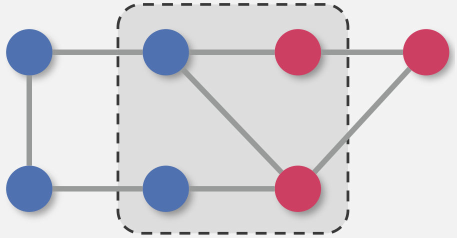

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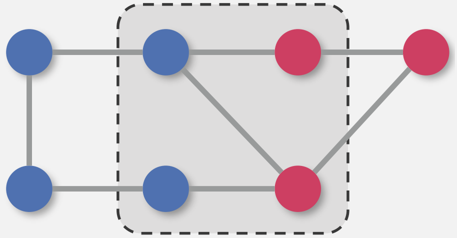
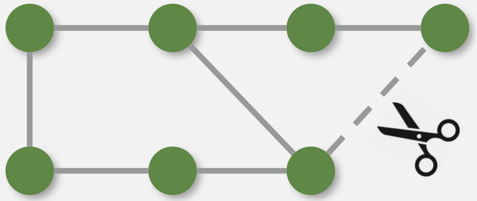


Practical Application

WIS: simple & efficient edge sparsification algorithm that outperforms alternative methods



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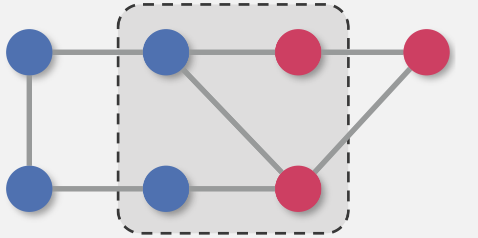

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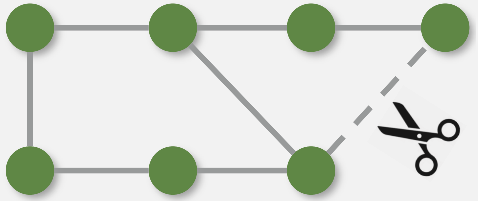

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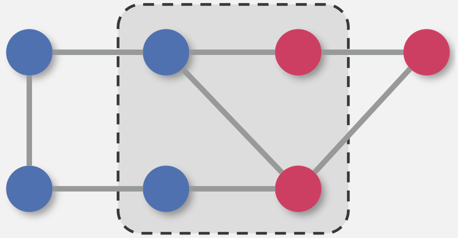

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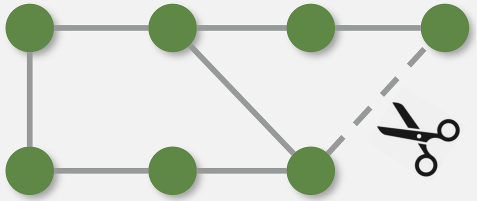

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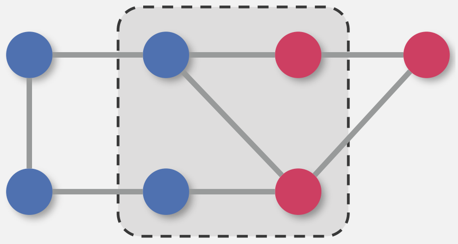

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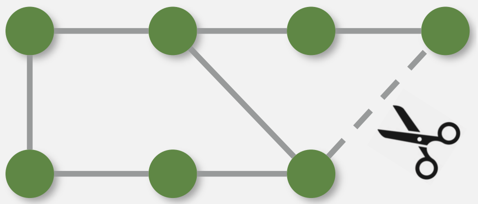

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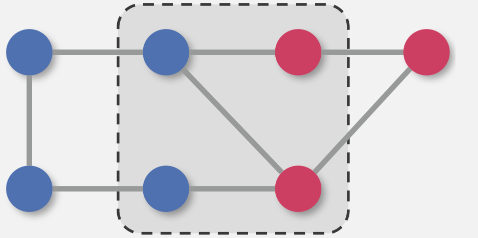

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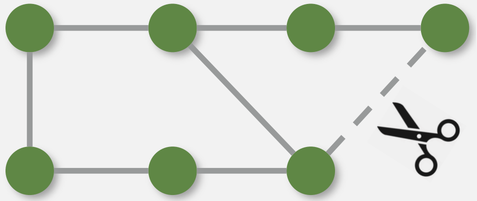

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