Machine learning detects terminal singularities

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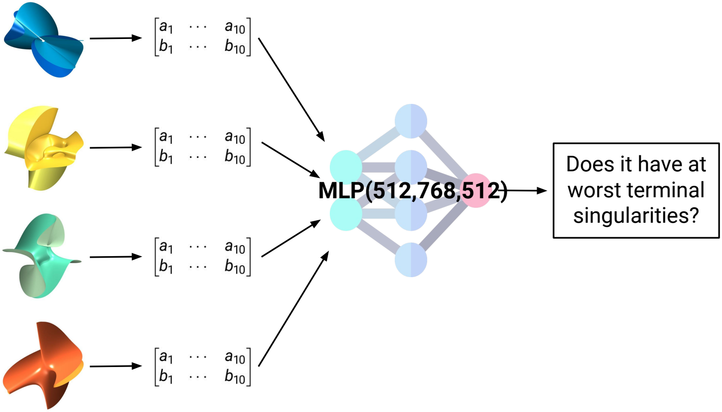
Machine Learning for Mathematics

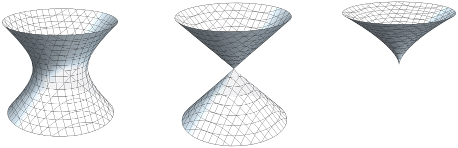
Proposal

An *AI-assisted workflow* for mathematical problems that are **unapproachable** with traditional methods.



The Mathematical Objects

Algebraic geometry is the study of shapes defined by solutions to systems of polynomial equations. They can be **smooth** or **have singularities**.



(a) $x^2 + y^2 = z^2 + 1$

(b) $x^2 + y^2 = z^2$

(c) $x^2 + y^2 = z^3$

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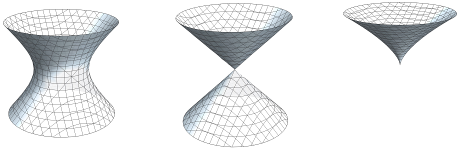

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$\mathbb{Q}$-Fano varieties are the ‘atoms’ of geometry

They are positively curved shapes with $\mathbb{Q}$-factorial terminal singularities. Their classification (still open!) is like building a **Periodic Table** for geometry.

Vectorisation

A 2×10 integer-valued matrix

$$\begin{bmatrix} a_1 & \cdots & a_{10} \\ b_1 & \cdots & b_{10} \end{bmatrix}$$

represents $\mathbb{C}^{10}$ with these points identified

$$(z_1, \ldots, z_{10}) \sim (\lambda^{a_1} \mu^{b_1} z_1, \ldots, \lambda^{a_{10}} \mu^{b_{10}} z_{10}) \, .$$

for any $\lambda, \mu \neq 0$. This is a *toric Fano variety* of

- rank **two** (# rows),
- dimension **eight** (# columns - # rows).

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Why? To make it challenging

- There is already a fast criterion for rank one.
- In low dimensions the problem is easier.

Symmetries

$$(1,2) \star M = \begin{bmatrix} a_2 & a_1 & \cdots & a_{10} \\ b_2 & b_1 & \cdots & b_{10} \end{bmatrix}$$

$\Longleftarrow$ S_{10} $\Longrightarrow$

$$M = \begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ b_1 & b_2 & \cdots & b_{10} \end{bmatrix}$$

$\Lleftarrow$ $\mathrm{GL}_2(\mathbb{Z})$ $\Rrightarrow$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot M = \begin{bmatrix} b_1 & b_2 & \cdots & b_{10} \\ a_1 & a_2 & \cdots & a_{10} \end{bmatrix}$$

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The standard form

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ 0 & b_2 & \cdots & b_{10} \end{bmatrix}$$

with $a_i, b_i \in \mathbb{Z}_{\geq 0}$, $a_{10} < b_{10}$, and the columns cyclically ordered.

Consequences of the ML Model

The model

A fully connected feedforward neural network predicts terminality with **95% accuracy**. It

- **inspires** a new algorithm to test terminality for toric Fanos.
- allows the **exploration** of the toric $\mathbb{Q}$-Fano landscape.

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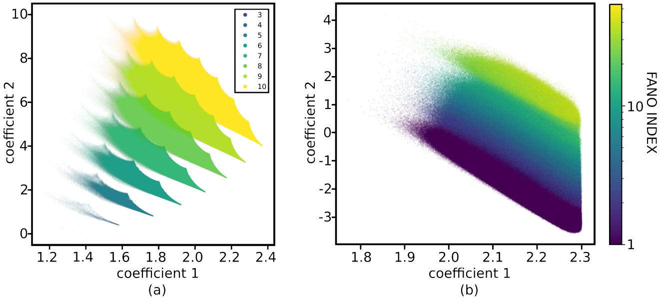
# Samples	Original Alg	New Alg	ML Model
1	1x	15x	450x
10 000	1x	15x	30 000x
100M $\mathbb{Q}$-Fano	**300 CPUyrs**	20 CPUyrs	**120 CPUhrs**

Sketching the Landscape

We visualise $\mathbb{Q}$-Fanos in $\mathbb{R}^2$ using the growth coefficients of their **quantum period**, a conjectured complete invariant. This would have been **impossible without an ML approach**.



(a) $\mathbb{Q}$-Fano varieties with rank one; (b) probable $\mathbb{Q}$-Fano varieties in dimension eight and rank two, coloured by Fano index.