# Many-body Approximation for Non-negative Tensors

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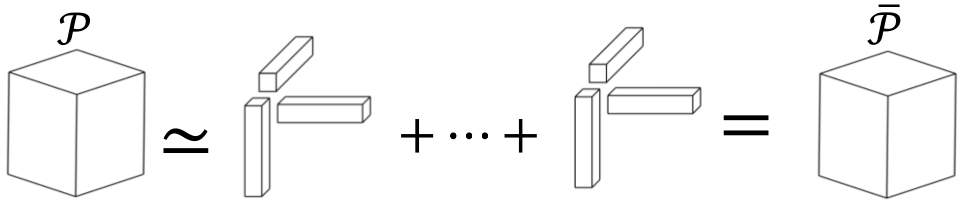
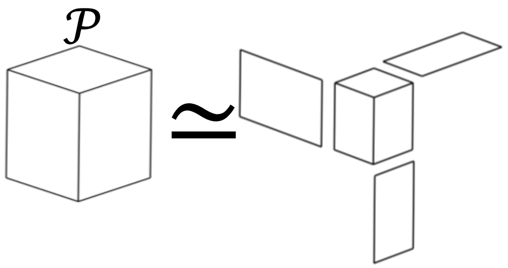
# Difficulties in tensor factorization

## ■ Model selection is not intuitive.

CP decomp.

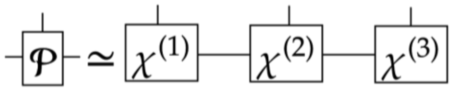



Tucker decomp.

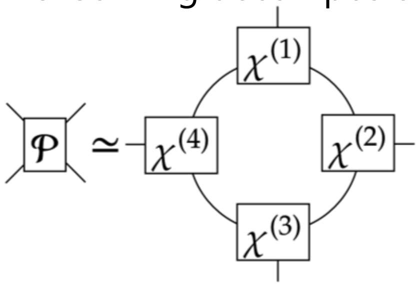



Decomp. with tensor networks

Tensor Train decomposion




Tensor ring decomposion



# Difficulties in tensor factorization

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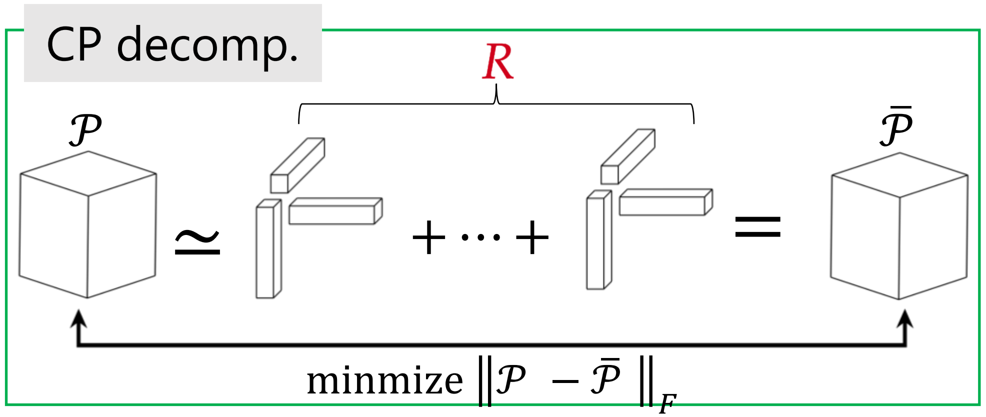
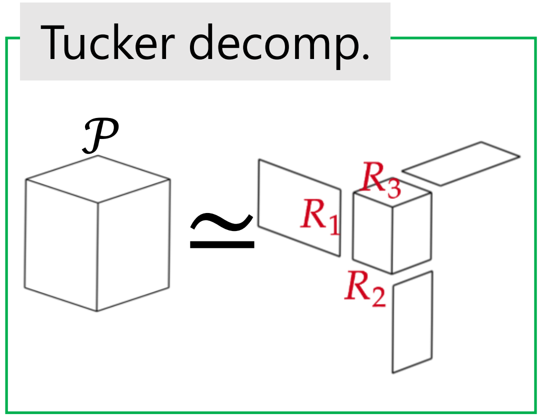
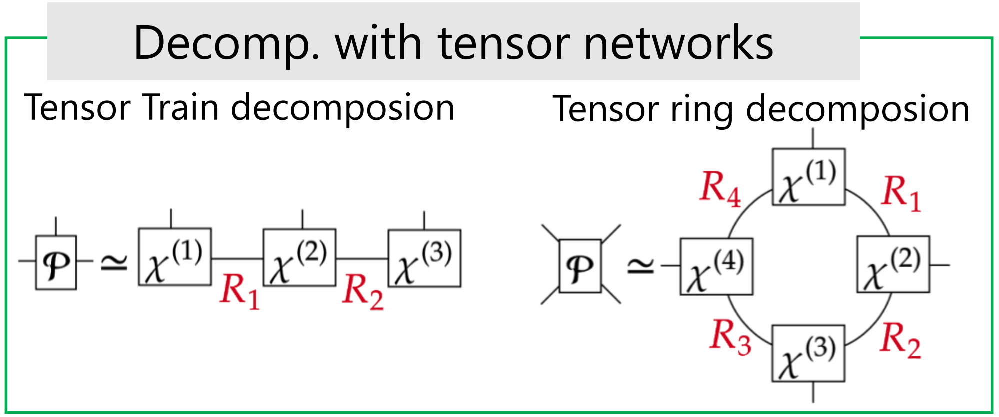


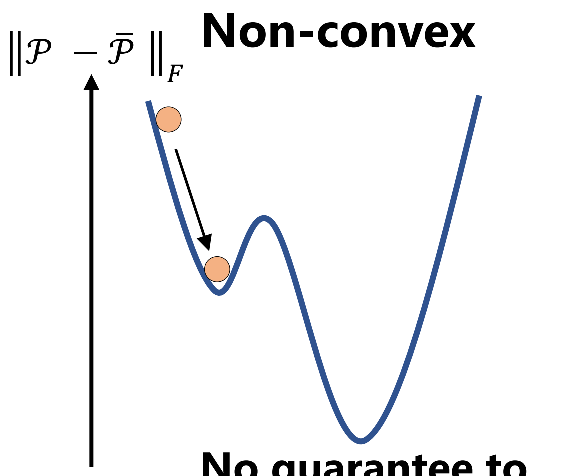




## ■ Optimization is difficult.

😟 **The objective function is typically non-convex.**
- Initial values dependency

😟 **Solution often might be indeterminate.**




**A convex, stable and intuitive tensor factorization is desired.**

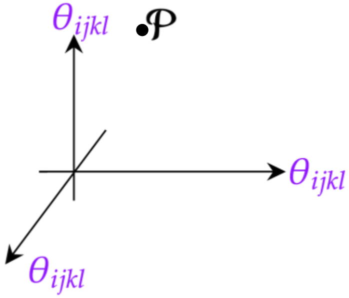
# Many-body approximation for non-negative tensors

**Energy function**

$$\mathcal{P}_{ijkl} = \frac{1}{Z}\exp[-E_{\theta}(i,j,k,l)]$$

Natural parameter
of exponential distribution family.

$$\sum_{ijkl} \mathcal{P}_{ijkl} = 1$$



# Many-body approximation for non-negative tensors

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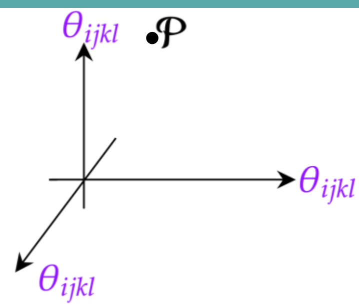


$$= \frac{1}{Z}\exp\left[E_i^{(1)} + \ldots + E_l^{(4)} + E_{ij}^{(12)} + \ldots + E_{kl}^{(34)} + E_{ijk}^{(123)} + \ldots + E_{jkl}^{(234)} + E_{ijkl}^{(1234)}\right]$$

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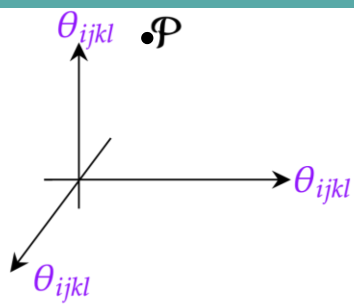


$$= \frac{1}{Z}\exp\left[E^{(1)}_{i} + \ldots + E^{(4)}_{l} + E^{(12)}_{ij} + \ldots + \boxed{E^{(34)}_{kl}} + E^{(123)}_{ijk} + \ldots + \boxed{E^{(234)}_{jkl}} + E^{(1234)}_{ijkl}\right]$$

Control relation between
mode-*k* and mode-*l*.

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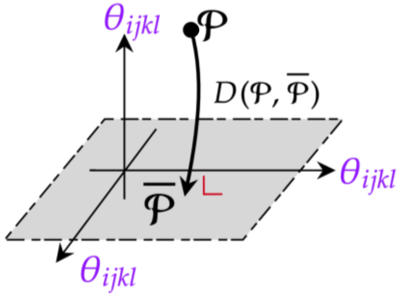


**One-body approx.**

$$\overline{\mathcal{P}}_{ijkl} = p_i^{(1)} p_j^{(2)} p_k^{(3)} p_l^{(4)}$$

**Rank-1 approximation**
**（mean-field approximation）**

［**NeurIPS 2021** Ghalamkari, K., Sugiyama, M. ］

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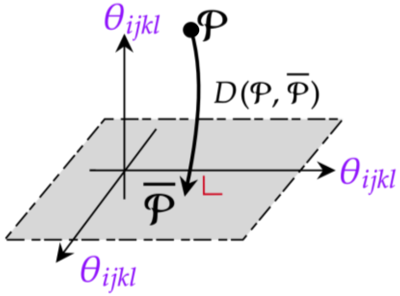


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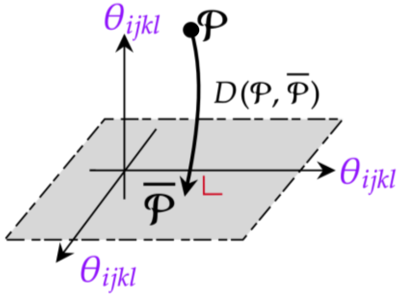
**Two-body approx.**

$$\overline{\mathcal{P}}_{ijkl} = X_{ij}^{(12)} X_{ik}^{(13)} X_{il}^{(14)} X_{jk}^{(23)} X_{jl}^{(24)} X_{kl}^{(34)}$$

**Rank-1 approximation**
**(mean-field approximation)**
[NeurIPS 2021 Ghalamkari, K., Sugiyama, M. ]

Larger Capability

# Many-body approximation for non-negative tensors

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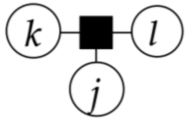

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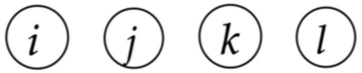

**Rank-1 approximation**
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[NeurIPS 2021 Ghalamkari, K., Sugiyama, M. ]

**Two-body approx.**

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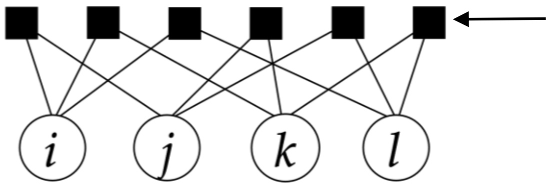
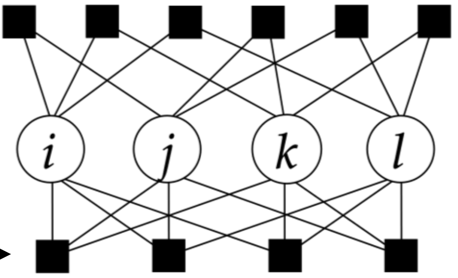


Two-body Interaction

**Three-body approx.**

$$\overline{\mathcal{P}}_{ijkl} = \chi_{ijk}^{(123)} \chi_{ijl}^{(124)} \chi_{ikl}^{(134)} \chi_{jkl}^{(234)}$$



Three-body Interaction

**Intuitive modeling focusing on interactions between modes**

Larger Capability

**The global optimal solution $\overline{\mathcal{P}}$ minimizing KL divergence from $\mathcal{P}$ can be obtained by a convex optimization.**

# Theoretical idea behind proposal

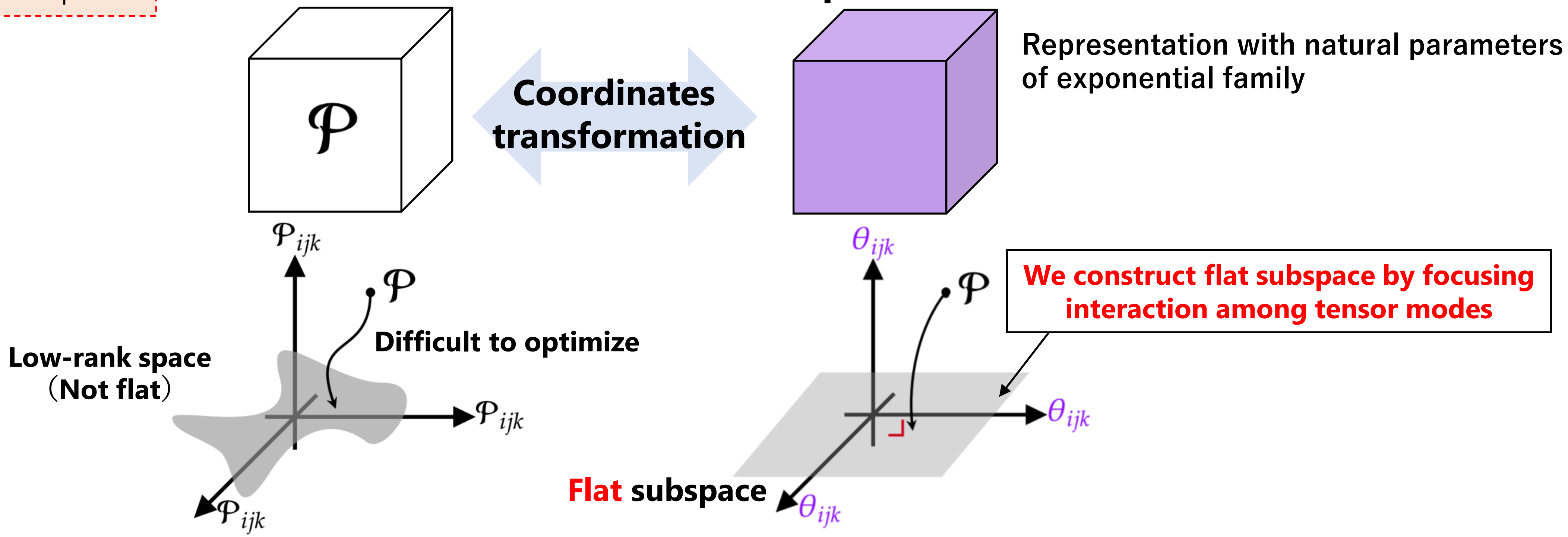
Index is discrete random variable

$$\sum_{ijk} \mathcal{P}_{ijk} = 1, \quad (i, j, k) \in \Omega = \{(1,1,1), \ldots, (I, J, K)\}$$

- **We regard a normalized tensor $\mathcal{P}$ as a discrete joint probability distribution whose sample space is an index set**
- **We use information geometry to formulate factorization as convex problem**

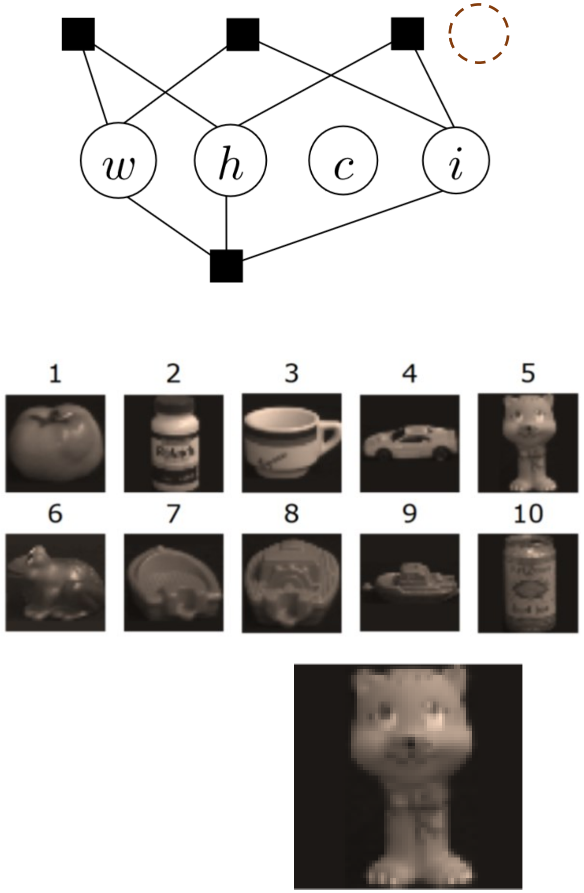
Geometry of $\boldsymbol{\theta}$-space

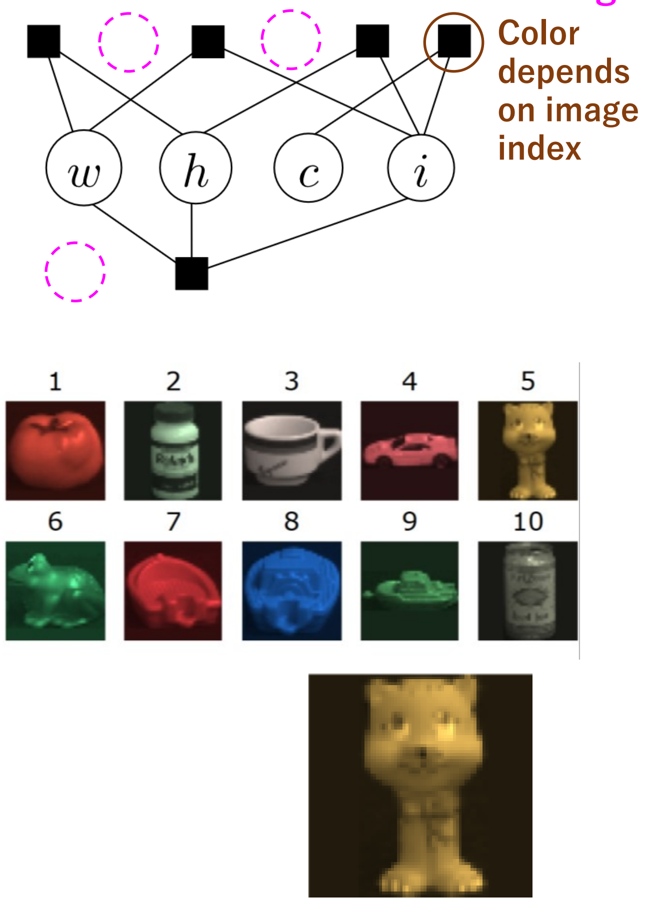
## $\boldsymbol{\theta}$- Representation



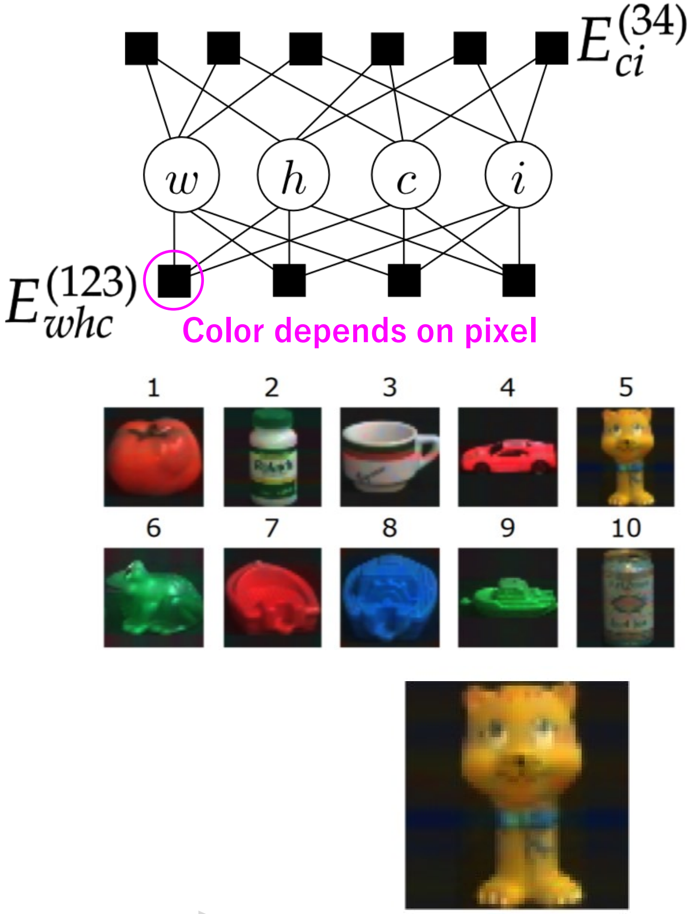

**Describing tensor factorization in $\boldsymbol{\theta}$-coordinate system makes it convex problem**

# Reconstruction for 40 × 40 × 3 × 10 tensor
(width, height, colors, # images)




**Three-body Approx.**




**Larger**
**ity**

**Intuitive model design that captures the relationship between modes**

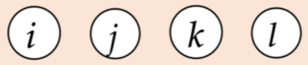
# Rank-free convex nonnegative tensor factorization

## Many-body Approximation

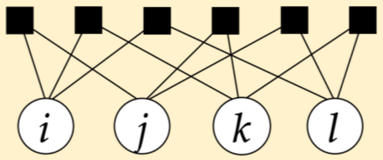
$$\mathcal{P}_{ijkl} = \exp\left[\sum_{i'=1}^{i}\sum_{j'=1}^{j}\sum_{k'=1}^{k}\sum_{l'=1}^{l}\theta_{i'j'k'l'}\right]$$

$$= \frac{1}{Z}\exp\left[E_i^{(1)}+\ldots+E_l^{(4)}+E_{ij}^{(12)}+\ldots+E_{kl}^{(34)}+E_{ijk}^{(123)}+\ldots+E_{jkl}^{(234)}+E_{ijkl}^{(1234)}\right]$$

One-body Approx.




Two-body Approx.

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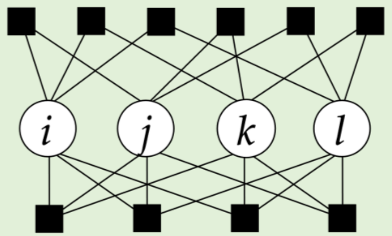


- **Convex optimization always provide unique solution**
- **More intuitive design than rank tuning**