Score-Based Generative Models with Lévy Processes

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MOTIVATION

Investigating score-based generative models **beyond Gaussian for noise injection** is an open question.

Property	Brownian motion	Isotropic α-stable Lévy process
Heavy-tailed	x	o
Continuous path	o	x
Density function	Exact	Not exact

} Slow convergence

Mode-collapse issue

Easy theoretical handling

Hard theoretical handling

MOTIVATION

Question: Are there any generative models using an alternative noise to overcome the intrinsic limitation of diffusion models?

Challenge

1) Common theoretical techniques based on Brownian motion may not be applicable.
2) The density function of the Lévy process has not an exact form.

CONTRIBUTION

- We propose a novel Score-based generative model, Lévy-Itô Model (LIM), which utilizes **isotropic α-stable Lévy processes** as noise injection.

- We derive **an exact reverse-time stochastic differential equation** driven by the Lévy process

- We derive **a fractional score function** to match the drift term of time-reversal SDEs and propose **fractional Denoising Score Matching**.

BACKGROUND

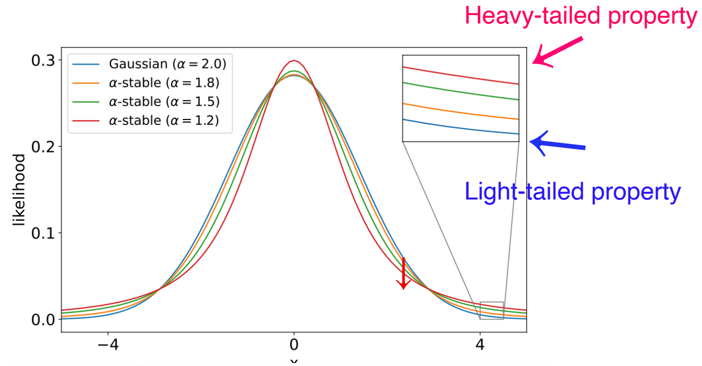
Isotropic α-stable distribution

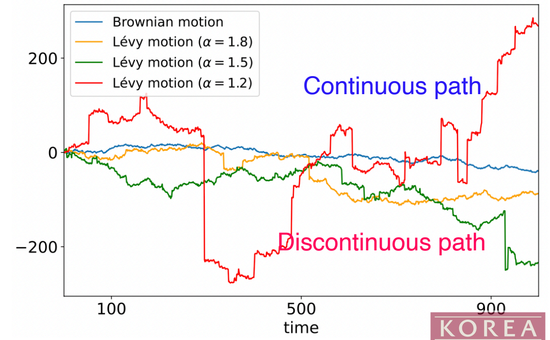
- $\alpha \in (0,2]$ be a characteristic exponent
- $\gamma \geq 0$ be a scale parameter
- 1-dimensional symmetric α-stable distribution $\mathcal{S}\alpha\mathcal{S}(\gamma)$

1) $X \sim \mathcal{S}\alpha\mathcal{S}(\gamma)$ then $\mathbb{E}[e^{i\langle \mathbf{u},\mathbf{x}\rangle}] = e^{-\gamma^{\alpha}||\mathbf{u}||^{\alpha}}$

2) Heavy-tail propoerties $P(X > \mathbf{x}) \sim ||\mathbf{x}||^{-\alpha}$

3) $\alpha = 2$; Gaussian $\alpha = 1$; Cauchy





BACKGROUND

Lévy processes

A stochastic process L_t is called **Lévy process** if

(i) L_t has independent increments

(ii) L_t has stationary increments

(iii) L_t is stochastically continuous.

} Minimal requirement for noise

If for all $s < t$, $(L_t - L_s) \overset{d}{=} L_{t-s}$ follows $\mathcal{S}\alpha\mathcal{S}^d((t-s)^{1/\alpha})$, where $\overset{d}{=}$ means that the two processes have the same law, then the Lévy process

L_t^α is called **isotropic α-stable Lévy process.**

THEORY

Time-reversal SDE driven by isotropic α-stable Lévy process

$$d\overleftarrow{X}_t = \left(-\frac{\beta(t)}{\alpha}\overleftarrow{X}_{t+} - \alpha \cdot \beta(t) \cdot S_t^{(\alpha)}(\overleftarrow{X}_{t+})\right)d\bar{t} + (\beta(t))^{\frac{1}{\alpha}} d\bar{L}_t^{\alpha}$$

↓ Fractional Score function

$$S_t^{(\alpha)}(\mathbf{x}) := \frac{\Delta^{\frac{\alpha-2}{2}} \nabla p_t(\mathbf{x})}{p_t(\mathbf{x})}$$

- $\Delta^{\frac{\beta}{2}}$: Fractional Laplacian of order $\frac{\beta}{2}$ for $\beta \in (-1,2)$
- $d\bar{t}$: Infinitesimal negative timestep
- $\bar{L}_t^{\alpha}$: Isotropic α-stable Lévy process such that time flows backward

THEORY

Variant of Euler-Maruyama with dynamic time increment

$$\mathbf{x}_t = \frac{a(t)}{a(s)}\mathbf{x}_s + \alpha^2\left(\frac{a(t)}{a(s)} - 1\right)S_s^{(\alpha)}(\mathbf{x}_s) + \left(\left(\frac{a(t)}{a(s)}\right)^{\alpha} - 1\right)^{\frac{1}{\alpha}}\epsilon$$

$$a(t) = \exp\left(-\int_0^t \frac{\beta(s)}{\alpha}ds\right), \quad \epsilon \sim \mathcal{S}\alpha\mathcal{S}^d(1)$$

Proof

$$d\overleftarrow{X}_t = \left(-\frac{\beta(t)}{\alpha}\overleftarrow{X}_{t+} - \alpha \cdot \beta(t) \cdot S_t^{(\alpha)}(\overleftarrow{X}_{t+})\right)d\bar{t} + (\beta(t))^{\frac{1}{\alpha}}d\bar{L}_t^{\alpha}$$

Using Semi-linear structure → Applying Itô-formula

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THEORY

Fractional Score Matching

$$L_1(\theta) = \mathbb{E}_{t,\mathbf{x}_t\sim p_t(\mathbf{x}_t)}\|\mathbf{s}_t(\mathbf{x}_t;\theta) - S_t^{(\alpha)}(\mathbf{x}_t)\|_2^2$$

$\mathbf{s}_t(\mathbf{x}_t,\theta)$ Fractional Score model

↕ Equivalent

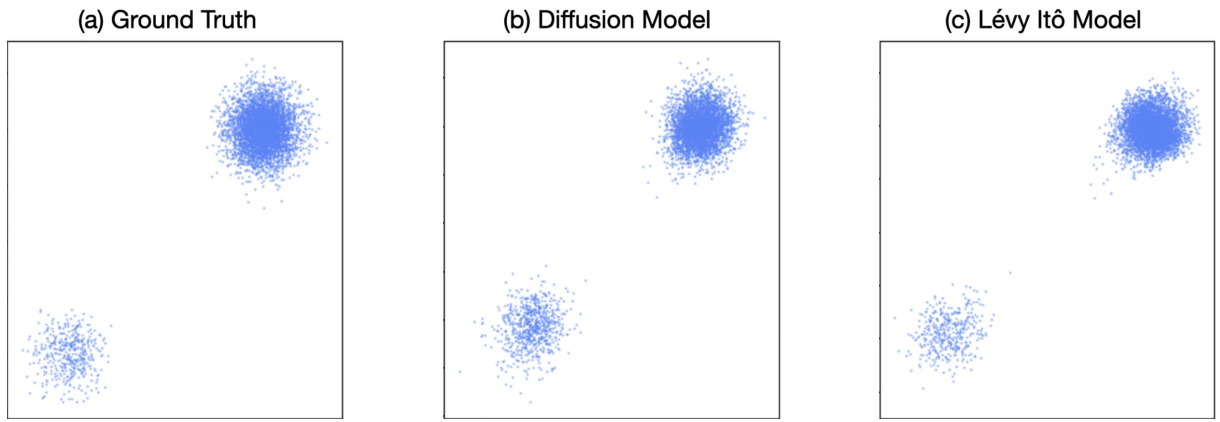
$$L_2(\theta) = \mathbb{E}_{t,(\mathbf{x}_0,\mathbf{x}_t)\sim p_t(\mathbf{x}_0,\mathbf{x}_t)}\|\mathbf{s}_t(\mathbf{x}_t;\theta) - S_t^{(\alpha)}(\mathbf{x}_t\,|\,\mathbf{x}_0)\|_2^2$$

$$S_t^{(\alpha)}(\mathbf{x}_t\,|\,\mathbf{x}_0) = \frac{\Delta^{\frac{\alpha-2}{2}}\nabla p_t(\mathbf{x}_t\,|\,\mathbf{x}_0)}{p_t(\mathbf{x}_t\,|\,\mathbf{x}_0)} = -\frac{1}{\gamma^{\alpha-1}(t)}\cdot\frac{1}{\alpha}\cdot\left(\frac{\mathbf{x}_t - a(t)\mathbf{x}_0}{\gamma(t)}\right)$$

Linear form
→ Easy to train!

RESULT

Good Mode estimation: Two mixture of Gaussian



Metric	Diffusion model	LIM
FID (↓)	8.312 ± 0.904	$\mathbf{0.663 \pm 0.376}$
MMD (↑)	0.025 ± 0.003	$\mathbf{0.02 \pm 0.002}$

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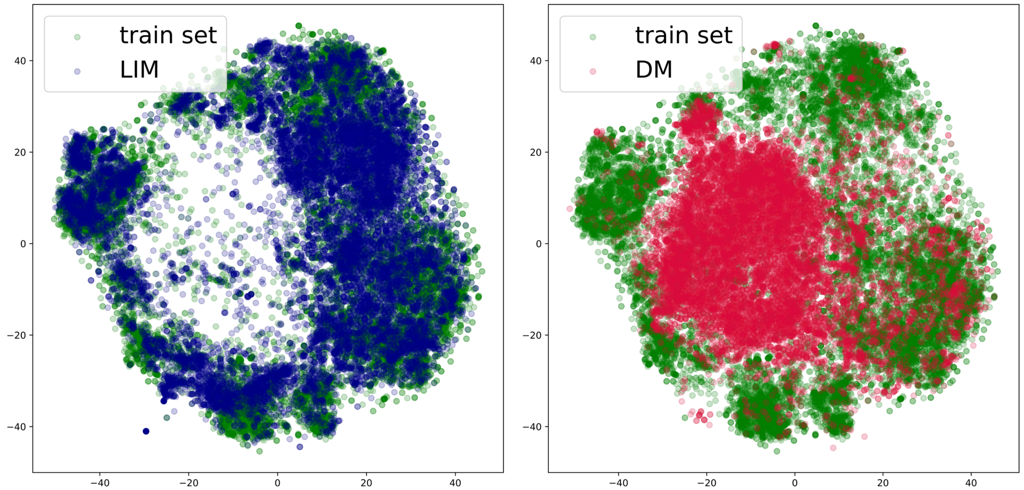
RESULT

Good Mode estimation: Imbalanced CIFAR 10



Metric	LIM	DM
FID↓	**21.07**	62.62
Recall↑	**0.5549**	0.5002
MMD↓	**0.00416**	0.01396

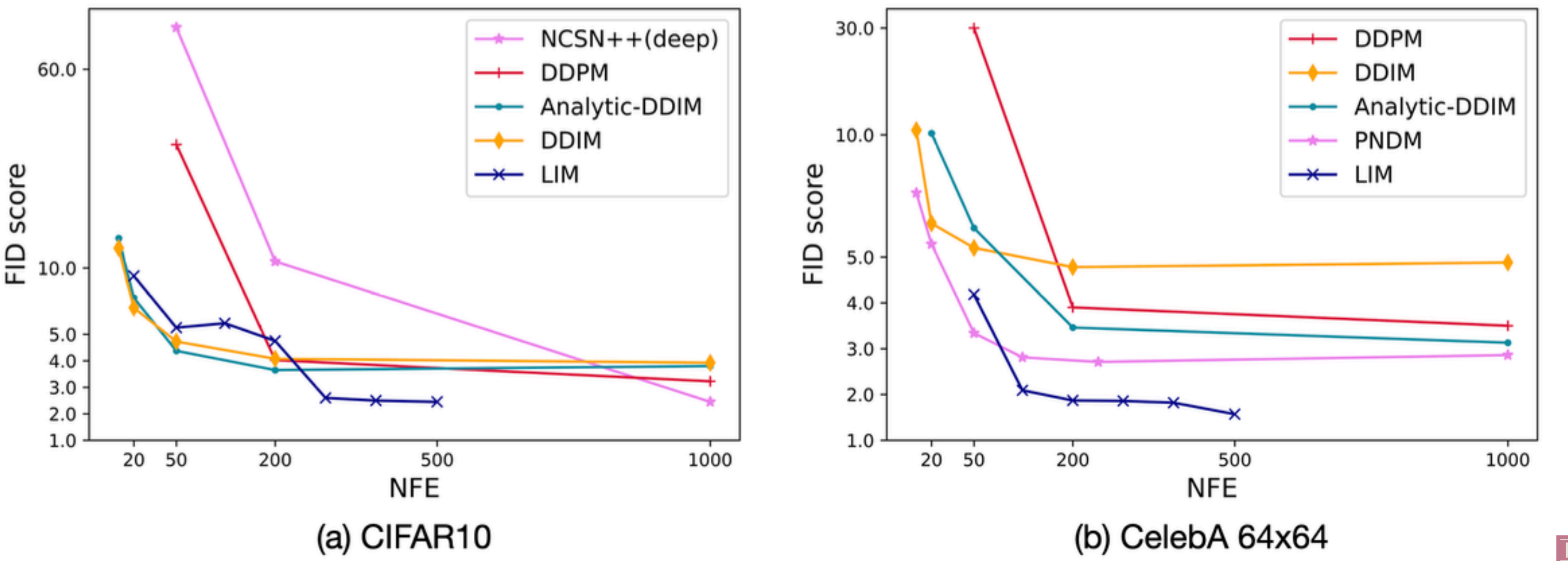
RESULT

Diverse sample generation

Recall (↑)	CIFAR10	CelebA	ImageNet
Diffusion model	0.6860	0.6437	0.6932
LIM	**0.6960**	**0.7007**	**0.6937**

RESULT

Fast Convergence rate



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RESULT

Comparable sample quality

FID (↓)	CIFAR10	CelebA	ImageNet
Diffusion model	2.44	2.23	14.23
LIM	2.44	1.57	12.97

CelebA-HQ

LUSN-Bedroom

LUSN-Church